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ABSTRACT

An investigation is made of electron behavior in a weakly ionized gas in a magnetic field and an arbitrarily time-dependent electric field. This problem is of interest both in astrophysics and the study of the basic phenomena in gas discharge. The electron distribution function study is based on the Boltzmann equation, which takes only the electron-atom collisions into account. A "moment method" is devised to solve the case wherein the electron collision frequency is uniform. In the subsequent analysis, the electron distribution function is first expanded in terms of both the "moments" and the associated Laguerre polynomials; a generating equation of the moments is then derived. This generating equation, which is linear and of first order in time, can be integrated readily.

The persistent solution is obtained in closed form for the quasisteady case. In the case of a high-frequency oscillating electric field, the drift velocity, the mean energy, the conductivity, and the dielectric constant are discussed.

I. INTRODUCTION

The present investigation concerns a slightly ionized gas in the presence of a uniform magnetic field and a time-dependent electrical field. The plasma is postulated to be homogeneous and, before the external fields are imposed, to possess Maxwellian distribution corresponding to the gas temperature T. The problem under study is the determination of the electron velocity distribution function of such a plasma, as well as the corresponding average electron energy and drift velocity. The Boltzmann equation for the present problem may be simplified by dropping the collision integrals describing the electron-electron and electron-ion encounters. In the following discussion, only the electron-atom collisions, for which the "conventional" close binary collision model may be assumed valid, are considered. Again, it is assumed that most of the collisions are elastic. This implies that the average electron energy is low and the external field is not very strong.

In the following analysis the fields generated by the internal distributions and motions of the charged particles are postulated to be negligible compared to the applied fields. Hence in the Boltzmann equation both $\mathbf{E}$ and $\mathbf{B}$ are considered to be known quantities, and hereafter we will write the electric field $\mathbf{E}$ in the following form:

$$\mathbf{E} = \mathbf{E}_0 \phi(t)$$

where $\phi(t)$ is any given function of time which is bounded and continuous at all time.

With the initial Maxwellian distribution we may obtain a solution for the subsequent time ($t > 0$) based on the assumption that the anisotropic part of the distribution function is a small perturbation of the isotropic distribution. In the following discussion a "moment method," which will be demonstrated later, is used. The main interest is to investigate the "persistent solution" ($t \gg \tau$, where τ is the mean collision time). A closed form solution for the quasisteady case (or some other special cases, e.g., the ac field) can be found by considering the asymptotic behavior of the distribution function. The analysis of this part is displayed in Section IV. The calculations for the electron drift velocity, mean energy, conductivity, and dielectric constant based on the "moments" are given in Section V.

II. DEDUCTION OF THE GOVERNING INTEGRO-DIFFERENTIAL EQUATION

Under the assumptions stated previously, the electron distribution function $f(\mathbf{v}, t)$ is considered to satisfy the Boltzmann equation of the following form (Chapman and Cowling 1952):

$$\left[\frac{\partial}{\partial t} - \frac{e}{m}\left(\mathbf{E}_0 \phi(t) + \mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial}{\partial \mathbf{v}}\right] f(\mathbf{v}, t) = \int_{v_m} \int_{\Omega} [f'_m f' - f_m f]\, g \sigma_m \, d\Omega \, d\mathbf{v} \tag{1}$$

In (1), the primes denote quantities after collisions, and $\sigma_m (g, \psi)$ is the differential cross section for elastic scattering of electron and neutral particles through an angle ψ. The solid angle $d\Omega$ may be expressed in the form

$$d\Omega = \sin \psi \, d\psi \, d\phi$$

where ϕ is the polar angle measured in a plane normal to the vector $\mathbf{g}$, which is the velocity of the electron relative to the colliding atom, i.e.,

$$\mathbf{g} = \mathbf{v} - \mathbf{v}_m$$

Furthermore, the subscript m denotes quantities which belong to the neutral atoms.

In order to solve (1) for $f(\mathbf{v}, t)$, we shall postulate that the electron distribution function may be written in the following form:

$$f(v, t) = f^{(0)}(v, t) + \mathbf{E}_0 \cdot \mathbf{v} f^{(1)}(v, t) + (\mathbf{B} \times \mathbf{E}_0) \cdot \mathbf{v} f^{(2)}(v, t) + (\mathbf{B} \cdot \mathbf{v})(\mathbf{B} \cdot \mathbf{E}_0)\, f^{(3)}(v, t) \tag{2}$$

The initial conditions of the unknown functions $f^{(0)}$, $f^{(1)}$, $f^{(2)}$, and $f^{(3)}$ may be given as follows:

$$\begin{aligned} \text{At } t = 0 \quad f^{(0)}(v, 0) &= N \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) \\ f^{(1)}(v, 0) &= 0 \\ f^{(2)}(v, 0) &= 0 \\ f^{(3)}(v, 0) &= 0 \end{aligned} \tag{3}$$

where N is the electron number density and k is the Boltzmann constant.

The expression (2) is merely the first two terms of the expansion of $f(\mathbf{v}, t)$ in terms of the spherical harmonic in the component of $\mathbf{v}$. A brief discussion of this expression is given by Wu (1961), from which we obtain

$$\frac{\partial f^{(0)}}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left[\frac{e}{m}\frac{v^3}{3}\left(f^{(1)} + B^2\cos^2\gamma f^{(3)}\right)E_0^2\phi(t) + \left(\frac{m}{M}\frac{v^4}{l}f^{(0)} + \frac{kT}{M}\frac{v^3}{l}\frac{\partial f^{(0)}}{\partial v}\right)\right] \tag{4}$$

$$\frac{\partial f^{(1)}}{\partial t} = -\frac{v}{l}f^{(1)} + \frac{e}{m}\left[\frac{1}{v}\frac{\partial f^{(0)}}{\partial v}\phi(t) - B^2 f^{(2)}\right] \tag{5}$$

$$\frac{\partial f^{(2)}}{\partial t} = -\frac{v}{l}f^{(2)} + \frac{e}{m}f^{(1)} \tag{6}$$

$$\frac{\partial f^{(3)}}{\partial t} = -\frac{v}{l}f^{(3)} + \frac{e}{m}f^{(2)} \tag{7}$$

where m and M are the electronic and atomic mass respectively, l the mean free path of electrons, and γ the angle between the field vectors $\mathbf{B}$ and $\mathbf{E}_0$.

Now we shall introduce the Laplace transform

$$\overline{f^{(i)}}(v, s) = \int_0^\infty f^{(i)} e^{-st}\,dt \qquad (i = 1,\ 2,\ \text{and}\ 3) \tag{8}$$

where s is the variable in the transformed space and has the property $Re[s] > 0$.

Using the initial conditions (3), we may transform (5), (6), and (7) to the following forms

$$\overline{f^{(1)}} = \frac{\tau}{(s\tau + 1)}\frac{e}{m}\left\{\frac{1}{v}L\left(\frac{\partial f^{(0)}}{\partial v}\phi\right) - B^2\overline{f^{(2)}}\right\} \tag{9}$$

$$\overline{f^{(2)}} = \frac{\tau}{(s\tau + 1)}\frac{e}{m}\overline{f^{(1)}} \tag{10}$$

$$\overline{f^{(3)}} = \frac{\tau}{(s\tau + 1)}\frac{e}{m}\overline{f^{(2)}} \tag{11}$$

where $L\,[(\partial f^{(0)}/\partial v)\phi]$ denotes the Laplace transform of $(\partial f^{(0)}/\partial v)\phi(t)$ and, for simplicity, we have denoted $l/v = \tau$, where τ is the collision time and, in general, is a function of v.

From (9), (10), and (11) we obtain

$$\overline{f^{(1)}} = \frac{(s\tau + 1)\tau}{(s\tau + 1)^2 + \omega^2\tau^2}\,\frac{e}{m}\,\frac{1}{v}\,L\left(\frac{\partial f^{(0)}}{\partial v}\,\phi\right) \tag{12}$$

$$\overline{f^{(2)}} = \frac{\tau^2}{(s\tau + 1)^2 + \omega^2\tau^2}\,\frac{e^2}{m^2}\,\frac{1}{v}\,L\left(\frac{\partial f^{(0)}}{\partial v}\,\phi\right) \tag{13}$$

$$\overline{f^{(3)}} = \frac{\tau^2}{(s\tau + 1)^2 + \omega^2\tau^2}\,\frac{e^3}{m^3}\,\frac{1}{v}\,L\left(\frac{\partial f^{(0)}}{\partial v}\,\phi\right) \tag{14}$$

where $\omega = (eB/m)$ is the electron cyclotron frequency. From (12) and (14), one may easily show that

$$f^{(1)} + B^2\cos^2\gamma f^{(3)} = \frac{e}{mv}\int_0^t e^{-\nu(t-t')}\left[\cos^2\gamma + \sin^2\gamma\cos\omega(t-t')\right] \times \frac{\partial f^{(0)}(v,t')}{\partial v}\,\phi(t')\,dt' \tag{15}$$

where $\nu = 1/\tau$.

Combining (15) and (4), we obtain an integro-differential equation for $f^{(0)}$ for $t \geq 0$.

$$\frac{\partial f^{(0)}}{\partial t} = \frac{1}{v^2}\,\frac{\partial}{\partial v}\left\{\frac{e^2E_0^2\phi(t)v}{3m^2}\int_0^t e^{-\nu(t-t')}\left[\cos^2\gamma + \sin^2\gamma\cos\omega(t-t')\right]\right.$$
$$\left. \times\,\frac{\partial f^{(0)}(v,t')}{\partial v}\,\phi(t')\,dt' + \frac{m\nu}{M}\left[v^3 f^{(0)} + \frac{mv^2}{4kT}\,\frac{\partial f^{(0)}}{\partial v}\right]\right\} \tag{16}$$

For simplicity hereafter we denote

$$u = \frac{v}{a}$$

$$a = \sqrt{\frac{2kT}{m}}$$

$$\epsilon = \frac{m}{M}$$

$$g(u, t) = f^{(0)}(v, t)$$

Then (16) becomes

$$u^2 \frac{\partial g}{\partial t} = \frac{\partial}{\partial u} \left\{ \frac{e^2 E_0^2}{6\,mkT} u^2 \phi(t) \int_0^t e^{-\nu(t-t')} [\cos^2\gamma + \sin^2\gamma \cos\omega(t-t')] \right.$$

$$\left. \times \frac{\partial g(u, t')}{\partial u} \phi(t')\, dt' + \epsilon\nu \left[u^3 g(u, t) + \frac{1}{2} u^2 \frac{\partial g}{\partial u} \right] \right\} \tag{17}$$

Two points should be remarked before this section is concluded:

1. It is possible to derive a general expression of the distribution function $f(\mathbf{v}, t)$ in terms of $f^{(0)}(v, t)$ or $g(u, t)$. From (12), (13), and (14), we may obtain the inverse transform as follows:

$$f^{(1)}(v, t) = \frac{e}{mv} \int_0^t e^{-\nu(t-t')} \cos\omega(t-t') \frac{\partial f^{(0)}(v, t')}{\partial v} \phi(t')\, dt' \tag{18}$$

$$f^{(2)}(v, t) = \frac{e^2}{m^2 v\omega} \int_0^t e^{-\nu(t-t')} \sin\omega(t-t') \frac{\partial f^{(0)}(v, t')}{\partial v} \phi(t')\, dt' \tag{19}$$

$$f^{(3)}(v, t) = \frac{e^3}{m^3 v\omega^2} \int_0^t e^{-\nu(t-t')} [1 - \cos\omega(t-t')] \frac{\partial f^{(0)}}{\partial v} \phi(t')\, dt' \tag{20}$$

Hence

$$f(\mathbf{v}, t) = g(u, t) + \frac{e}{ma^2 u} \int_0^t e^{-\nu(t-t')} \left\{ (\mathbf{E}_0 \cdot \mathbf{u}) \cos\omega(t-t') + \frac{e(\mathbf{B} \times \mathbf{E}_0) \cdot u}{m\omega} \sin\omega(t-t') \right.$$

$$\left. + \frac{e^2}{m^2\omega^2} (\mathbf{B} \cdot \mathbf{E}_0)(\mathbf{B} \cdot \mathbf{u}) [1 - \cos\omega(t-t')] \right\} \frac{\partial g}{\partial u} \phi(t')\, dt' \tag{21}$$

Therefore, once $g(u, t)$ is known, the distribution function $f(\mathbf{v}, t)$ may be immediately calculated from (21).

2. The asymptotic expression of $f(\mathbf{v}, t)$ as $t \to \infty$ may be obtained by showing, from Eq. (18), that:

$$f^{(1)}(v, t) = \frac{e}{mau} \frac{\int_0^t e^{\nu t'} \cos \omega (t - t') \frac{\partial f^{(0)}}{\partial v} \phi(t')\, dt'}{e^{\nu t}}$$

$$= \frac{e}{2mau} \frac{\int_0^t e^{(\nu - i\omega) t'} \frac{\partial f^{(0)}}{\partial v} \phi(t')\, dt'}{e^{(\nu - i\omega)t}} + \frac{\int_0^t e^{(\nu + i\omega) t'} \frac{\partial f^{(0)}}{\partial v} \phi(t')\, dt'}{e^{(\nu + i\omega)t}}$$

$$\xrightarrow[t \to \infty]{} \frac{e}{2ma^2u} \sum_{k=0}^{\infty} (-1)^k \frac{\partial^k}{\partial t^k} \left(\frac{\partial g_\infty}{\partial u} \phi(t) \right) \left[\frac{1}{(\nu - i\omega)^{k+1}} + \frac{1}{(\nu + i\omega)^{k+1}} \right]$$

$$= \frac{e}{ma^2u} \sum_{k=0}^{\infty} (-1)^k \frac{\partial^k}{\partial t^k} \left(\frac{\partial g_\infty}{\partial u} \phi(t) \right) Re \left(\frac{1}{\nu + i\omega} \right)^{k+1} = f_\infty^{(1)} \tag{22}$$

where $g_\infty(t)$ represents the behavior of g as $t \to \infty$. Similarly, from (19) and (20):

$$f^{(2)}(v, t) \xrightarrow[t \to \infty]{} f_\infty^{(2)}(v, t) = \frac{e^2}{m^2 a^2 u \omega} \sum_{k=0}^{\infty} (-1)^k \frac{\partial^k}{\partial t^k} \left(\frac{\partial g_\infty}{\partial u} \phi(t) \right) Im \left(\frac{1}{\nu + i\omega} \right)^{k+1} \tag{23}$$

$$f^{(3)}(v, t) \xrightarrow[t \to \infty]{} f_\infty^{(3)}(v, t) = \frac{e^3}{m^3 a^2 u \omega^2} \sum_{k=0}^{\infty} (-1)^k \frac{\partial^k}{\partial t^k} \left(\frac{\partial g_\infty}{\partial u} \phi(t) \right) \left[\frac{1}{\nu^{k+1}} - Re \left(\frac{1}{\nu + i\omega} \right)^{k+1} \right] \tag{24}$$

Hence

$$f(\mathbf{v}, t) \xrightarrow[t \to \infty]{} f_\infty(\mathbf{v}, t) = g_\infty(u, t) + \frac{e}{mau} \sum_{k=0}^{\infty} (-1)^k \frac{\partial^k}{\partial t^k} \left(\frac{\partial g_\infty}{\partial u} \phi(t) \right) \Bigg\{ (\mathbf{E}_0 \cdot \mathbf{u})\, Re \left(\frac{1}{\nu + i\omega} \right)^{k+1}$$

$$+ \frac{e}{m\omega} (\mathbf{B} \times \mathbf{E}_0) \cdot \mathbf{u}\, Im \left(\frac{1}{\nu + i\omega} \right)^{k+1} + \frac{e^2}{m^2 \omega^2} (\mathbf{B} \cdot \mathbf{E}_0)(\mathbf{B} \cdot \mathbf{u}) \times \left[\frac{1}{\nu^{k+1}} - Re \left(\frac{1}{\nu + i\omega} \right)^{k+1} \right] \Bigg\} \tag{25}$$

III. METHOD OF SOLUTION

A. Expansion of the Function $g(u, t)$

In an attempt to solve (17), we shall devise a method which contains two essential steps: (1) expanding the function $g(u, t)$ in terms of its moments as defined in the following:

$$M_n = 2\int_0^\infty g(u, t)\, u^{n+2}\, du \tag{26}$$

and (2) then deriving the "generating equation" for these moments.

First, we shall expand the function $g(u, t)$ in terms of the Laguerre polynomial

$$g(u, t) = e^{-\lambda u^2} \sum_{k=0}^{\infty} a_k(\lambda, t)\, L_k^{1/2}(\lambda^2 u^2) \tag{27}$$

where λ is a parameter which may be a function of t. The determination of $\lambda(t)$ will be discussed later.

The Laguerre polynomial $L_k^{1/2}$ is defined (Magnus and Oberhettinger 1954) as

$$L_k^{1/2}(\lambda^2 u^2) = \sum_{n=0}^{k} \binom{k+(1/2)}{k-n} \frac{(-\lambda^2 u^2)^n}{n!} \tag{28}$$

The reason for choosing the Laguerre polynomial of order 1/2 is based mainly on its orthogonality property. The discussion of this choice may be found in an Appendix, in which it has also been shown that the coefficients a_k may be expressed in terms of the moments:

$$a_k = \sum_{n=0}^{k} \binom{k}{n} \frac{(-1)^n \lambda^{2n+3}}{\left(n + \frac{1}{2}\right)!} M_{2n}(t) \tag{29}$$

The choice of λ will be based on the criterion that the fastest convergence of (27) may be obtained.

To illustrate this, we consider the special case at t = 0. Since

$$g(u,0) = e^{-\lambda^2 u^2} \sum_{k=0}^{\infty} a_k(\lambda,0)\, L_k^{1/2}(\lambda^2 u^2) = N(\sqrt{\pi}\, a)^{-3} e^{-u^2}$$

it may be shown that

$$a_k(\lambda,0)\, \frac{1}{2}!\, \binom{k+(1/2)}{k} = N(\sqrt{\pi} a)^{-3} \sum_{n=0}^{k} \frac{(-1)^n \left(k-\frac{1}{2}\right)!\, \lambda^{2n+3}}{(k-n)!\, n!} \tag{30}$$

Therefore,

$$g(u,0) = N(\sqrt{\pi}\, a)^{-3} e^{-\lambda^2 u^2} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} (-1)^n \lambda^{2n+3} L_k^{1/2}(\lambda^2 u^2)$$

$$= N(\sqrt{\pi}\, a)^{-3} e^{-\lambda^2 u^2} \lambda^3 \sum_{k=0}^{\infty} (1-\lambda^2)^k L_k^{1/2}(\lambda^2 u^2) \tag{31}$$

Letting $\lambda = 1$, we have the first term only. This implies that the fastest convergence may be achieved if $\lambda = 1$. However, when $t \neq 1$ this is no longer true, as will be shown in Section IV.

B. The Moment Equations

In order to determine the moments, we shall return to (17). In the following discussion, we shall restrict ourselves to the case that ν is uniform; in some cases this furnishes a good approximation for weakly ionized gas (Delcroix 1960). In this case, if each term of (17) is multiplied by u^{2n+2} and integrated over all values of u, we obtain a system of integro-differential-difference equations which may be written as follows:

$$\frac{dM_{2n+2}}{dt} + \epsilon\nu(2n+2) M_{2n+2} = (2n+2)(2n+3) \left\{ \frac{e^2 E_0^2}{6\, mkT} \phi(t) \int_0^t e^{-\nu(t-t')} \right.$$

$$\left. \times \left[\cos^2\gamma + \sin^2\gamma \cos\omega(t-t')\right] \phi(t') M_{2n}(t')\, dt' + \frac{1}{2} \epsilon\nu M_{2n}(t) \right\} \tag{32}$$

Since, by definition,

$$M_0 = 2\int_0^\infty g(u,t)\,u^2\,du = 2\int_0^\infty g(u,0)\,u^2\,du$$

$$= 2\int_0^\infty N(\sqrt{\pi}\,a)^{-3}\exp(-u^2)\,u^2\,du = \frac{N}{2\pi a^3} \tag{33}$$

we may therefore proceed to calculate the higher moments M_2, M_4, etc., by integrating (32). At this point, in principle, the problem is solved.

In closing this section, we may remark that it is possible to express the distribution function $f(\mathbf{v}, t)$ in terms of the moments,

$$f(\mathbf{v},t) = \sum_{k=0}^{\infty}\sum_{n=0}^{k}\Bigg\{\binom{k}{n}\frac{(-1)^n\lambda^{2n+3}}{\left(n+\frac{1}{2}\right)!}M_{2n}(t)\,e^{-\lambda^2u^2}L_k^{1/2}(\lambda^2u^2)$$

$$-\frac{e}{mu}\int_0^t e^{-\nu(t-t')}\,Y(\mathbf{u},t)\,e^{-\lambda^2u^2}\frac{(-1)^n\,2u\,\lambda^{2n+5}}{\left(n+\frac{1}{2}\right)!}M_{2n}(t)$$

$$\times\left[\binom{k}{n}L_k^{1/2}(\lambda^2u^2) + \binom{k+1}{n}L_k^{3/2}(\lambda^2u^2)\right]\phi(t')\,dt'\Bigg\} \tag{34}$$

where

$$Y(\mathbf{u},t) = \left[(\mathbf{E}_0\cdot\mathbf{u})\cos\omega(t-t') + \frac{e}{m\omega}(\mathbf{B}\times\mathbf{E}_0)\cdot\mathbf{u}\sin\omega(t-t') + \frac{e^2}{m^2\omega^2}(BE_0\cos\gamma)(\mathbf{B}\cdot\mathbf{u})(1-\cos\omega(t-t')\right]$$

IV. PERSISTENT SOLUTIONS IN THE CASE OF UNIFORM COLLISION FREQUENCY

It is seen from the previous discussion that considerable mathematical simplification may be obtained if the collision frequency may be considered to be independent of u. From (32), we may obtain by integration

$$M_{2n+2}(t) = e^{-\epsilon\nu(2n+2)t}\left[M_{2n+2}(0) + \int_0^t e^{\epsilon\nu(2n+2)t'}\,\Phi_{2n}(t')\,dt'\right] \tag{35}$$

where

$$\Phi_{2n}(t) = (2n+2)(2n+3)\left\{\frac{e^2E_0^2}{6\,mkT}\,\phi(t)\int_0^t e^{-\nu(t-t')}\left[\cos^2\gamma + \sin^2\gamma\cos\omega(t-t')\right]\phi(t')M_{2n}(t')dt' + \frac{1}{2}\,\epsilon\nu M_{2n}(t)\right\}$$

In (35) the integration constant $M_{2n+2}(0)$ may easily be evaluated:

$$M_{2n+2}(0) = 2\int_0^\infty g(u,0)\,u^{2n+4}du$$

$$= N(\sqrt{\pi}\,a)^{-3}\,\Gamma\left(n + \frac{5}{2}\right) \tag{36}$$

If we denote $M_{2n}^\infty(t)$ as the asymptotic form of $M_{2n}(t)$, i.e., $M_{2n}(t) \xrightarrow[t\to\infty]{} M_{2n}^\infty(t)$, then we have

$$M_{2n+2}^\infty(t) = \sum_{k=0}^\infty (-1)^k \frac{1}{[\epsilon\nu(2n+2)]^{k+1}}\,\frac{\partial^k \Phi_{2n}^\infty}{\partial t^k} \tag{37}$$

$$\Phi_{2n}^\infty = \frac{(2n+2)(2n+3)\,e^2E_0^2\,\phi(t)}{6\,mkT}\sum_{p=0}^\infty (-1)^p$$

$$\times\left[\frac{\cos^2\gamma}{\nu^{p+1}} + \sin^2\gamma\,Re\left(\frac{1}{\nu+i\omega}\right)^{p+1}\right]\sum_{s=0}^{p}\binom{p}{s}\phi^{(s)}M_{2n}^{\infty\,(p-s)} + \frac{1}{2}\,\epsilon\nu M_{2n}^\infty(t) \tag{38}$$

Possible further simplifications of (37) and (38) may be found for the following special cases:

A. Quasisteady Case

The quasisteady case is defined when the following conditions hold true

$$\frac{\phi^{(k)}}{\nu^k} << \phi, \qquad \frac{M_{2n}^{(k)}}{\nu^k} << M_{2n} \qquad k = 1, 2, 3, \cdots$$

where $\phi^{(k)}$ and $M_{2n}^{(k)}$ denote the derivatives of ϕ and M_{2n}, respectively. The physical meaning of these conditions is that the changes of the functions ϕ and M_n for the time interval $\tau (= 1/\nu)$ are small compared to the values of the functions. This situation enables us to rewrite (37) and (38) as follows:

$$\Phi_{2n}^{\infty} = \frac{(2n + 2)(2n + 3)\, e^2 E_0^2 \phi(t)}{6\, mkT} \left[\frac{\cos^2 \gamma}{\nu} + \sin^2 \gamma \, Re \left(\frac{1}{\nu + i} \right) \right] \phi M_{2n}^{\infty} + \frac{1}{2} \epsilon \nu M_{2n}^{\infty}(t) \tag{39}$$

$$M_{2n+2}^{\infty} = \frac{1}{\epsilon \nu (2n + 2)} \Phi_{2n}^{\infty} \tag{40}$$

If we further introduce

$$M_{2n+2}^{\infty *} = \frac{1}{\left(n + \frac{3}{2} \right)!} M_{2n+2}^{\infty}$$

$$M_{2n}^{\infty *} = \frac{1}{\left(n + \frac{1}{2} \right)!} M_{2n}^{\infty} \tag{41}$$

$$\cdots\cdots$$

one may deduce from (39), (40), and (41) that

$$M_{2n+2}^{\infty *} = \left[\frac{e^2 E^2 \phi^2(t)}{3\, m\, \epsilon kT} \left(\frac{\cos^2 \gamma}{\nu^2} + \frac{\sin^2 \gamma}{\nu^2 + \omega^2} \right) + 1 \right]^{n+1} M_0^{\infty *} \tag{42}$$

where

$$M_0^{\infty *} = M_0^*(0) = N (\sqrt{\pi} a)^{-3}$$

Hence, the asymptotic form of $g(u, t)$ for large time may be given in a rather simple form:

$$g^{\infty}(u, t) = N(\sqrt{\pi}a)^{-3} \lambda_{\infty}^{3} e^{-\lambda_{\infty}^{2} u^{2}} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} (-1)^{n} \lambda_{\infty}^{2n} [1 + \Omega(t)]^{n} L_{k}^{1/2} (\lambda_{\infty}^{2} u^{2}) \tag{43}$$

where

$$\Omega(t) = \frac{e^{2} E_{0}^{2} \phi^{2}(t)}{3 m \epsilon k T} \left(\frac{\cos^{2} \gamma}{\nu^{2}} + \frac{\sin^{2} \gamma}{\nu^{2} + \omega^{2}} \right)$$

It is clear that the optimum choise of λ_{∞} is

$$\lambda_{\infty}(t) = [1 + \Omega(t)]^{-1/2}$$

Then

$$g^{\infty}(u, t) = N(\sqrt{\pi}a)^{-3} [1 + \Omega(t)]^{-3/2} \exp \left[- \frac{u^{2}}{1 + \Omega(t)} \right] \tag{44}$$

because

$$\sum_{k=0}^{\infty} \cdot \sum_{n=0}^{k} \binom{k}{n} (-1)^{n} L_{k}^{1/2} (\lambda_{\infty}^{2} u^{2}) = 1$$

As a consequence,

$$\frac{\partial g^{\infty}}{\partial u} = -2N(\sqrt{\pi}a)^{-3} u [1 + \Omega(t)]^{-3/2} \exp \left\{ - \frac{u^{2}}{1 + \Omega(t)} \right\} \tag{45}$$

With the substitution of (44) and (45) into (25), the persistent solution of $f(v, t)$ is found.

B. High-Frequency Oscillating Field

The second special case for which (37) and (38) may be simplified is that the electric field E varies according to a time function $\phi(t) = e^{i\beta t}$. In this case, the approximation used previously is no longer valid if the frequency β is of the same order of ν. However, simplification is still possible, since sometimes we need only the information of M_0 and M_2. To illustrate this point, discussions are given in the following section.

V. APPLICATIONS

The distribution function $f(\mathbf{v}, t)$, discussed previously, takes a rather complicated form. From a practical point of view this is not desirable. However, it should be noted that, in the calculation of the ensemble average of certain physical quantities (such as energy or velocity) it is really not necessary to first obtain the distribution function. Most of the calculations may be accomplished by direct considerations of the "moment." To demonstrate this, we shall consider the following examples.[1]

A. Electron Drift Velocity

By definition

$$< \mathbf{v} > = \frac{1}{N} \int_{-\infty}^{\infty} f(\mathbf{v}) \, \mathbf{v} \, d\mathbf{v}$$

$$= \frac{4\pi e a^3}{3Nm} \int_0^t e^{-\nu(t-t')} \Bigg\{ \cos \omega(t-t') \, \mathbf{E}_0 + \frac{e}{m\omega} (\mathbf{B} \times \mathbf{E}_0) \sin \omega(t-t')$$

$$+ \frac{e^2}{m^2 \omega^2} (\mathbf{B} \cdot \mathbf{E}_0) \mathbf{B} \left[1 - \cos \omega(t-t')\right] \Bigg\} \phi(t') \int_0^{\infty} \frac{\partial g}{\partial u} u^3 \, du \, dt' \tag{46}$$

Since

$$\int_0^{\infty} \frac{\partial g}{\partial u} u^3 du = - 3 \int_0^{\infty} u^2 g(u, t) \, du = - \frac{3}{2} M_0 = - \frac{3}{2} \left(\frac{N}{2\pi a^3} \right)$$

we thus have

$$< \mathbf{v} > = - \frac{e}{m} \int_0^t e^{-\nu(t-t')} \Bigg\{ \cos \omega(t-t') \, \mathbf{E}_0 + \frac{e}{m\omega} (\mathbf{B} \times \mathbf{E}_0) \sin \omega(t-t')$$

$$+ \frac{e^2}{m^2 \omega^2} (\mathbf{B} \cdot \mathbf{E}_0) \mathbf{B} \left[1 - \cos \omega(t-t')\right] \Bigg\} \phi(t') \, dt' \tag{47}$$

[1] The discussion is still restricted to the assumption of uniform ν.

This implies that, for any given function $\phi(t)$, the drift velocity may be readily computed. It should be remarked that if ν is not independent of u, (47) is, of course, not valid and should be modified to the form

$$\langle \mathbf{v} \rangle = -\frac{2e}{3m} \int_0^t \left\langle \frac{d}{du}\, u^3 e^{-\nu(t-t')} \right\rangle \left\{ \cos\omega(t-t')\,\mathbf{E}_0 + \frac{e}{m\omega}(\mathbf{B}\times\mathbf{E}_0)\sin\omega(t-t') + \frac{e^2}{m^2\omega^2}(\mathbf{B}\cdot\mathbf{E}_0)\mathbf{B}\left[1-\cos\omega(t-t')\right] \right\} \phi(t')\,dt' \tag{48}$$

The calculation of electron current density $\mathbf{J} = -en \langle \mathbf{v} \rangle$ for the case $\phi = \cos\beta t$ has been performed based on (47). For simplicity, we only list the results of the "persistent" part (large time, $t >> 1/\nu$)

$$J_{1\infty} = \frac{Ne^2E_0}{m}\left\{ \frac{[\nu(\nu^2+\beta^2+\omega^2)\cos\beta t + \beta(\nu^2+\beta^2-\omega^2)\sin\beta t](1-\cos^2\gamma)}{[\nu^2+(\beta-\omega)^2]\,[\nu^2+(\beta+\omega)^2]} + \frac{\cos^2\gamma\,[\nu\cos\beta t + \beta\sin\beta t]}{\nu^2+\beta^2} \right\} \tag{49}$$

$$J_{2\infty} = \frac{Ne^2E_0}{m}\sin\gamma \left\{ \frac{\omega(\nu^2+\omega^2-\beta^2)\cos\beta t + 2\nu\omega\beta\sin\beta t}{[\nu^2+(\beta-\omega)^2]\,[\nu^2+(\beta+\omega)^2} \right\} \tag{50}$$

$$J_{3\infty} = \frac{Ne^2E_0\sin\gamma\cos\gamma}{m}\left\{ \frac{\nu\cos\beta t + \beta\sin\beta t}{\nu^2+\beta^2} - \frac{[\nu(\nu^2+\omega^2+\beta^2)\cos\beta t + \beta(\beta^2-\omega^2+\nu^2)\sin\beta t]}{[\nu^2+(\beta-\omega)^2]\,[\nu^2+(\beta+\omega)^2]} \right\} \tag{51}$$

where $J_{1\infty}$, $J_{2\infty}$, and $J_{3\infty}$ are the three components of the current density J_∞ in the coordinate system specified in Fig. 1. The total current $\mathbf{J}$ is therefore

$$\mathbf{J} = J_1\mathbf{i}_1 + J_2\mathbf{i}_2 + J_3\mathbf{i}_3$$

where, furthermore, $\mathbf{J}$ may be considered as the sum of conduction and polarization currents. In other words (Stratton 1941)

$$\mathbf{J}_\infty = \mathbf{J}_e + \frac{\partial\mathbf{P}}{\partial t}$$

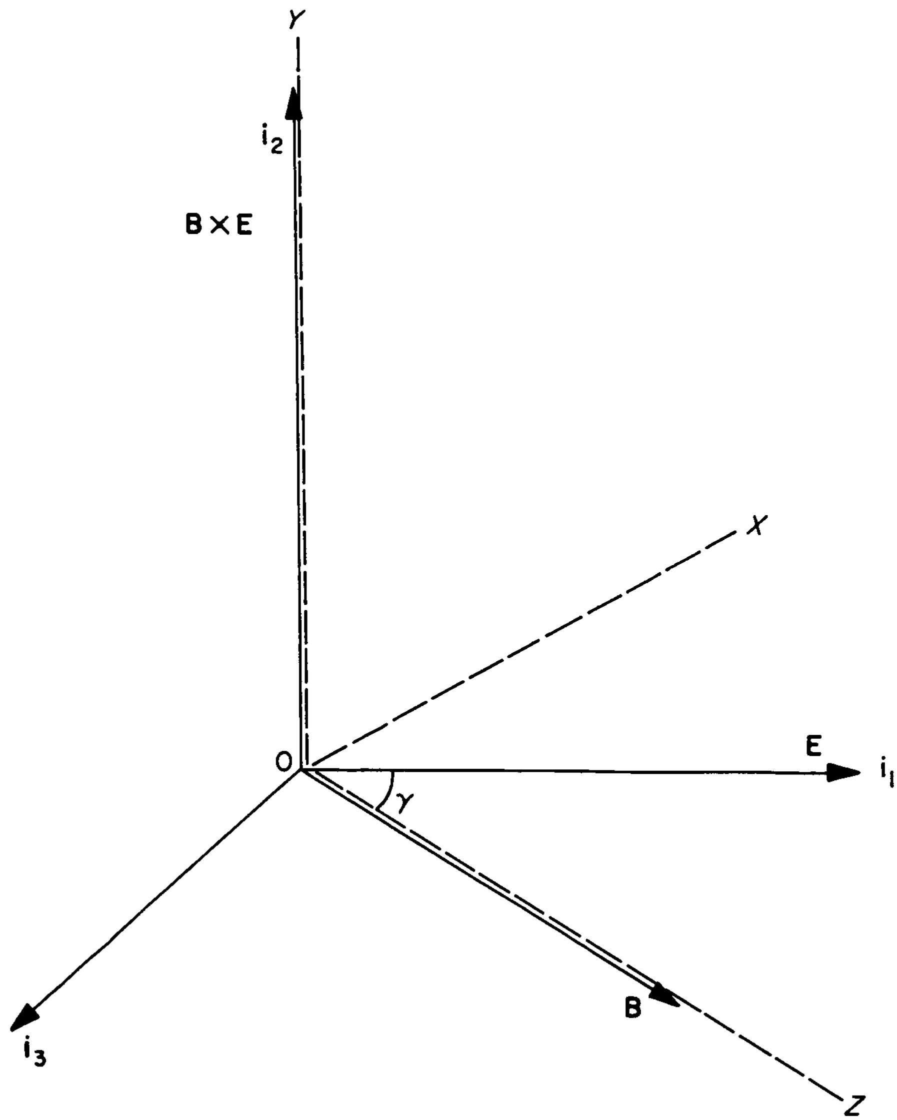

Fig. 1. Coordinates for the electron current density

where **P** is the polarization vector of the plasma

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E}$$

ϵ_0 being the dielectric constant in free space. In the most general case the actual dielectric constant ϵ is expected to be a tensor; we may therefore write

$$P_i = (\epsilon_{ik} - \epsilon_0 \, \delta_{ik}) E_k$$

where δ_{ik} is the Kronecker delta.

Postulating that ϵ_{ik} is independent of time, we have

$$\frac{\partial P_i}{\partial t} = (\epsilon_{ik} - \epsilon_0 \, \delta_{ik}) \frac{\partial E_k}{\partial t}$$

for $\phi = \cos \beta t$

$$\frac{\partial P_i}{\partial t} = -(\epsilon_{ik} - \epsilon_0 \, \delta_{ik}) \beta \sin \beta t \, E_{0k}$$

Hence

$$J_{\infty i} = \left[\sigma_{ik} \cos \beta t - (\epsilon_{ik} - \epsilon_0 \, \delta_{ik}) \beta \sin \beta t \right] E_{0k} \tag{52}$$

If the x, y, z system shown in Fig. 1 is used, then the tensorial conductivity and dielectric constant may be computed by comparing (52) with (49), (50), and (51), with the following results:

1. Conductivity

$$\sigma_{xx} = \sigma_{yy} = \frac{\sigma \nu^2}{2} \left[\frac{1}{\nu^2 + (\omega - \beta)^2} + \frac{1}{\nu^2 + (\omega + \beta)^2} \right]$$

$$\sigma_{yx} = -\sigma_{xy} = \frac{\sigma}{2} \nu \left[\frac{\omega - \beta}{\nu^2 + (\omega - \beta)^2} + \frac{\beta + \omega}{\nu^2 + (\omega + \beta)^2} \right]$$

$$\sigma_{zz} = \sigma \left[\frac{\nu^2}{\nu^2 + \beta^2} \right] \qquad \sigma_{zx} = \sigma_{zy} = 0 \qquad \sigma = \frac{Ne^2}{m\nu}$$

2. Dielectric constant

$$\epsilon_{xx} = \epsilon_{yy} = \epsilon_0 \left\{ 1 - \frac{\omega_p^2}{2\beta} \left[\frac{\beta - \omega}{\nu^2 + (\beta - \omega)^2} + \frac{\beta + \omega}{\nu^2 + (\beta + \omega)^2} \right] \right\}$$

$$\epsilon_{xy} = -\epsilon_{yx} = \epsilon_0 \frac{\omega_p^2}{2\beta} \left[\frac{\nu}{\nu^2 + (\beta - \omega)^2} - \frac{\nu}{\nu^2 + (\beta + \omega)^2} \right]$$

$$\epsilon_{zz} = \epsilon_0 \left(1 - \frac{\omega_p^2}{\nu^2 + \beta^2} \right), \qquad \epsilon_{zx} = \epsilon_{zy} = 0, \qquad \omega_p^2 = \frac{Ne^2}{m\epsilon_0}$$

These results agree with those given by Margenau (1946), Kelly (1960), and Ginzburg (1953).

B. Electron Mean Energy

The mean electron energy may be expressed in terms of the second moment, since

$$\frac{1}{2} m < v^2 > = \frac{2\pi m}{N} \int_0^\infty f^{(0)}(v, t)\, v^4\, dv = \frac{\pi m}{N} a^5 M_2^\infty(t) \tag{53}$$

However, from (37) and (38):

$$\Phi_0 = \frac{e^2 E_0^2 \phi(t) M_0}{mkT} \sum_{p=0}^{\infty} (-1)^p \left[\frac{\cos^2 \gamma}{\nu^{p+1}} + \sin^2 \gamma \, Re \left(\frac{1}{\nu + i\omega} \right)^{p+1} \right] \frac{d^p \phi(t)}{dt^p}$$

and

$$M_2^\infty = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2\epsilon\nu)^{k+1}} \frac{\partial^k \Phi_0^\infty(t)}{\partial t^k} \tag{54}$$

In the following, we shall consider the example $\phi = \cos \beta t$. The mean electron energy in this case may be determined by first finding $M_2^\infty(t)$. Since it can be readily shown that

$$\Phi_0^\infty(t) = \frac{M_0 e^2 E_0 \cos \beta t}{mkT} \left\{ \left[\cos^2 \gamma \left(\frac{\nu}{\nu^2 + \beta^2} \right) + \sin^2 \gamma \, Re \left(\frac{\nu + i\omega}{(\nu + i\omega)^2 + \beta^2} \right) \right] \cos \beta t \right.$$
$$\left. + \left[\cos^2 \gamma \left(\frac{\beta}{\nu^2 + \beta^2} \right) + \sin^2 \gamma \left(\frac{\beta}{\nu^2 + \beta^2} \right) + \sin^2 \gamma \, Re \left(\frac{\beta}{(\nu + i\omega)^2 + \beta^2} \right) \right] \sin \beta t \right\} + 3\epsilon\nu M_0 \tag{55}$$

$$M_2^\infty(t) = \frac{e^2 E_0^2 M_0}{2\epsilon\nu kTm} \left\{ \left[\cos^2 \gamma \left(\frac{\nu}{\nu^2 + \beta^2} \right) + \sin^2 \gamma \, Re \left(\frac{\nu + i\omega}{(\nu + i\omega)^2 + \beta^2} \right) \right] \right.$$
$$\times \left[\cos^2 \beta t + \frac{\beta}{\epsilon\nu} \cos \beta t \sin \beta t - \frac{\beta}{2\epsilon\nu} \left(\frac{\beta\epsilon\nu}{\epsilon^2\nu^2 + \beta^2} \cos 2\beta t + \frac{\beta^2}{\epsilon^2\nu^2 + \beta^2} \sin 2\beta t \right) \right]$$
$$\left. + \frac{1}{2} \left[\cos^2 \gamma \left(\frac{\beta}{\nu^2 + \beta^2} \right) + \sin^2 \gamma \, Re \left(\frac{\beta}{(\nu + i\omega)^2 + \beta^2} \right) \right] \left[\frac{\epsilon^2\nu^2 \sin 2\beta t}{\epsilon^2\nu^2 + \beta^2} - \frac{\beta\epsilon\nu \cos 2\beta t}{\epsilon^2\nu^2 + \beta^2} \right] \right\} + \frac{3}{2} M_0 \tag{56}$$

and

$$M_0 = \frac{N}{2\pi \left(\frac{2kT}{m} \right)^{3/2}}$$

we obtain

$$\frac{1}{2}\,m<v^2>=\frac{Me^2E_0^2}{2m^2\nu}\left\{\left[\cos^2\gamma\left(\frac{\nu}{\nu^2+\beta^2}\right)+\sin^2\gamma\,Re\left(\frac{\nu+i\omega}{(\nu+i\omega)^2+\beta^2}\right)\right]\right.$$

$$\times\left[\cos^2\beta t-\frac{\beta}{2\epsilon\nu}\left(\frac{\beta\epsilon\nu}{\epsilon^2\nu^2+\beta^2}\cos 2\beta t-\frac{\epsilon^2\nu^2}{\epsilon^2\nu^2+\beta^2}\sin 2\beta t\right)\right]+\frac{1}{2}$$

$$\left.\times\left[\cos^2\gamma\left(\frac{\beta}{\nu^2+\beta^2}\right)+\sin^2\gamma Re\left(\frac{\beta}{(\nu+i\omega)^2+\beta^2}\right)\right]\left[\frac{\epsilon^2\nu^2\sin 2\beta t}{\epsilon^2\nu^2+\beta^2}-\frac{\beta\epsilon\nu\cos 2\beta t}{\epsilon^2\nu^2+\beta^2}\right]\right\}+\frac{3}{2}\,kT \tag{57}$$

For the limiting case as $\beta \to 0$

$$\frac{1}{2}\,m<v^2>=\frac{Me^2E_0^2}{3m^2\nu^2}\left[\frac{\nu^2+\omega^2\cos^2\gamma}{\nu^2+\omega^2}\right]+\frac{3}{2}\,kT$$

This result agrees with that obtained previously (Wu 1961).

VI. DISCUSSION

In the discussion of the "persistent" solution, we have implicitly assumed that the time function $\phi(t)$ and its derivatives are continuous and bounded. This causes certain limitations to the application of the results. However, if one uses (35) instead of (37) and (38), the derivatives of the function $\phi(t)$ do not have to be continuous and bounded.

The discussion of the parameter λ, which appears in the expansion (27), is given for the quasisteady case. For large time ($t >> \tau$), it has been shown that, if we take

$$\lambda_\infty(t) = \left[1 + \frac{e^2 E_0^2 \phi(t)}{3 m \epsilon k T}\left(\frac{\cos^2\gamma}{\nu^2} + \frac{\sin^2\gamma}{\nu^2 + \beta^2}\right)\right]^{1/2} \tag{58}$$

the best convergence may be obtained. Although the expression of $\lambda_\infty(t)$ which may give a similar result still remains unknown, for the general case it is yet conceivable that (58) may be used to obtain good convergence of (27).

It may be remarked in closing that, for $\beta \to 0$, the drift velocity calculated by the present method agrees with the solution obtained from a generalized Spitzer's equation (Zmuidzinas and Wu 1960), as expected.

APPENDIX

Expansion of the Function $g(u, t)$

Because the function $g(u, t)$ is expected to have "Gaussian type" behavior, we propose the following expansion

$$g(u, t) = e^{-\lambda^2 u^2} \sum_{k=0}^{\infty} a_k(\lambda, t)\, L_k^{\alpha}(\lambda^2 u^2) \tag{A-1}$$

where λ is an arbitrary parameter discussed in the paper, $L_k^{\alpha}(\lambda^2 u^2)$ $(k = 1, 2, \cdots)$ are the associated Laguerre polynomials in which α is to be determined, and $a_k(a, t)$ are the expansion coefficients.

In an attempt to determine the value of α for the proper expansion, we consider the following integral

$$I \equiv \int_0^{\infty} g(u, t)\, L_{k'}^{\alpha}(\lambda^2 u^2)\, (\lambda^2 u^2)^{\alpha}\, 2\lambda^2 u\, du \tag{A-2}$$

$$= \int_0^{\infty} e^{-\lambda^2 u^2} \sum_{k=0}^{\infty} a_k(\lambda, t)\, L_k^{\alpha}(\lambda^2 u^2)\, L_{k'}^{\alpha}(\lambda^2 u^2)(\lambda^2 u^2)^{\alpha}\, d(\lambda^2 u^2)$$

$$= a_{k'}(\lambda, t)\, \alpha!\binom{k'+\alpha}{k'} \tag{A-3}$$

However, from the definition of the Laguerre polynomial, the integral I may also be written as

$$I = \int_0^{\infty} g(u, t) \sum_{n=0}^{k} \binom{k'+\alpha}{k'-n} \frac{(-\lambda^2 u^2)^n (\lambda^2 u^2)^{\alpha}}{n!}\, d(\lambda^2 u^2)$$

$$= 2 \sum_{n=0}^{k} \binom{k'+\alpha}{k'-n} \frac{(-1)^n \lambda^{2n+2\alpha+2}}{n!} \int_0^{\infty} g(u, t)\, u^{2n+2\alpha+1}\, du \tag{A-4}$$

It may be visualized that if we set $\alpha = 1/2$

$$I = \sum_{n=0}^{k'} \binom{k'+\frac{1}{2}}{k'-n} \frac{(-1)^n \lambda^{2n+3}}{n!} M_{2n}(t) \tag{A-5}$$

This enables us to express the coefficients a_k in terms of the moments:

$$a_k = \sum_{n=0}^{k} \binom{k}{n} \frac{(-1) \lambda^{2n+3}}{\left(n + \frac{1}{2}\right)!} M_{2n}(t) \tag{A-6}$$

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